

# A ZOLL COUNTEREXAMPLE TO A GEODESIC LENGTH CONJECTURE

FLORENT BALACHEFF<sup>1</sup>, CHRISTOPHER CROKE<sup>2</sup>,  
AND MIKHAIL G. KATZ<sup>3</sup>

ABSTRACT. We construct a counterexample to a conjectured inequality  $L \leq 2D$ , relating the diameter  $D$  and the least length  $L$  of a nontrivial closed geodesic, for a Riemannian metric on the 2-sphere. The construction relies on Guillemin's theorem concerning the existence of Zoll surfaces integrating an arbitrary infinitesimal odd deformation of the round metric. Thus the round metric is not optimal for the ratio  $L/D$ .

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## 1. ZOLL SURFACES AND GUILLEMIN DEFORMATION

Given a Riemannian metric on the 2-sphere, we consider its diameter  $D$  and the length  $L$  of its shortest nontrivial closed geodesic. The first inequality relating the two invariants was obtained by the second-mentioned author [Cr88], who proved the bound  $L \leq 9D$ . The constant in the inequality was successively improved by M. Maeda [Ma94], A. Nabutovsky and R. Rotman [NR02], and S. Sabourau [Sa04]. The best known bound is  $L \leq 4D$ . Nabutovsky and Rotman conjectured the inequality  $L \leq 2D$  [NR02, Introduction], meaning that the round

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metric of  $S^2$  is optimal for the relationship between these two invariants. We give a few examples of surfaces satisfying the case of equality  $L = 2D$ :

- (1) a surface of revolution in  $\mathbb{R}^3$  obtained from an ellipse with major axis on the  $x$ -axis;
- (2) a circular “pillow”, obtained by doubling the flat unit disk;
- (3) a more general pillow obtained by doubling the region enclosed by a closed curve of constant width in the plane;
- (4) rotationally invariant Zoll surfaces.

The existence of such diverse examples may have led one to expect that none of these metrics are optimal for the ratio  $L/D$ .

It turns out that a counterexample to the inequality  $L \leq 2D$  may be found among Zoll surfaces, namely surfaces all of whose geodesics are closed, and whose prime geodesics all have equal length  $2\pi$ . More precisely, while the rotationally symmetric Zoll surfaces do satisfy (the boundary case of equality of) the conjectured inequality, there exist other families of Zoll surfaces such that  $L > 2D$ . Such surfaces can be obtained as smooth variations of the round metric.

Let  $(S^2, g_0)$  be the 2-sphere endowed with the round metric. Denote by  $a : S^2 \rightarrow S^2$  its antipodal map. Let  $C_{\text{odd}}^\infty(S^2, \mathbb{R})$  be the space of smooth odd functions on  $S^2$ , i.e. smooth real valued functions  $f$  satisfying  $f \circ a = -f$ . The following existence theorem for Zoll surfaces is due to V. Guillemin [Gu76].

**Theorem 1.1** (Guillemin). *For every  $f \in C_{\text{odd}}^\infty(S^2, \mathbb{R})$ , there exists a smooth one-parameter family  $g_t = \Psi_t^f g_0$  of smooth Zoll metrics such that  $\Psi_0^f = 1$ , the conformal factor  $\Psi_t^f$  satisfies  $(d\Psi_t^f/dt)|_{t=0} = f$ , and all prime periodic geodesics of  $(S^2, g_t)$  have length  $2\pi$ .*

Note that this result is a converse to P. Funk’s theorem [Fu13], to the effect that a smooth variation  $g_t = \Phi_t g_0$  of the round metric by smooth Zoll metrics necessarily satisfies  $(d\Phi_t/dt)|_{t=0} \in C_{\text{odd}}^\infty(S^2, \mathbb{R})$ . A survey of Zoll surfaces appeared in [Bes78, Chapter 4], see also [LM02].

We exploit such Guillemin deformations to show that the round metric is not even a local maximum of the ratio  $L/D$  among Zoll surfaces. The precise statement of our result relies on the notion of a  $Y$ -like set.

**Definition 1.2.** A subset of the unit circle is called *Y-like* if it contains a triple of vectors  $\{u, v, w\}$  such that there exist positive real numbers  $a > 0, b > 0, c > 0$  satisfying  $au + bv + cw = 0$ . A subset of the unit tangent bundle  $US^2$  of  $S^2$  will be called *Y-like* if its intersection with the unit tangent vectors at  $p$  is *Y-like* for every  $p \in S^2$ .

Note that a subset of the unit circle is  $Y$ -like if and only if every open semicircle contains an element of the set.

We will denote by  $ds_0$  the element of length for the round metric  $g_0$  on the sphere. The notion of an amply negative function is motivated in Remark 2.4 below.

**Definition 1.3.** An odd function  $f$  is called *amply negative* if the set of unit tangent directions to great half-circles  $\tau$  satisfying  $\int_{\tau} f ds_0 < 0$ , is a Y-like subset of  $US^2$ .

**Theorem 1.4.** *If  $f$  is an amply negative function then the smooth variation  $\{g_t\} = \Psi_t^f g_0$  of the round metric  $g_0$  by smooth Zoll metrics satisfies  $L(g_t) > 2D(g_t)$  for sufficiently small  $t > 0$ .*

Combined with the existence of amply negative functions proved in Section 5, our theorem yields the desired counterexample.

These metrics also provide a counterexample to another conjecture of Nabutovsky and Rotman [NR07, Conjecture 1, p. 13]. Their conjecture would imply that for every point  $p$  of a closed Riemannian manifold  $(M, g)$ , there is a nontrivial geodesic loop at  $p$  of length at most  $2D(g)$ . Here a geodesic loop is a geodesic segment with identical endpoints. This conjecture is easily seen to be true for non-simply-connected manifolds, by exploiting non-contractible loops, cf. [Ka07]. In our examples, the shortest geodesic loop at every  $p$  has length  $2\pi$ , while the diameter is strictly smaller than  $\pi$ .

Sections 2 and 3 contain a proof of Theorem 1.4 modulo on the existence of amply negative functions. The existence of the latter is verified in Sections 4 and 5.

## 2. AMPLY NEGATIVE ODD FUNCTIONS

Our goal is to find amply negative functions  $f \in C_{\text{odd}}^{\infty}(S^2, \mathbb{R})$ , such that the corresponding Guillemin deformation  $g_t$  of the standard round metric  $g_0$  satisfies  $D(g_t) < \pi$  for  $t$  small enough (while all geodesics remain closed of length  $2\pi$ ). By the compactness of the unit tangent circle bundle  $US^2$ , we obtain the following lemma.

**Lemma 2.1.** *For every amply negative function  $f$  there is a constant  $\nu(f) > 0$  with the following property. For every  $(p, v) \in US^2$ , there is a great half-circle  $\tau$  issuing from  $p \in S^2$ , forming an acute angle with  $v$ , and satisfying  $\int_{\tau} f ds_0 < -\nu(f)$ .*

Denote by  $L_0$  the length functional with respect to the round metric  $g_0$ . Given a geodesic segment  $\gamma$  of length  $L_0(\gamma) < \pi$ , we denote by  $P_{\gamma}$  the 1-parameter family of piecewise geodesic paths with the following two properties:

- the path joins the endpoints of  $\gamma$ ,
- the path consists of a pair of imbedded geodesic segments of equal length.

Elements of  $P_\gamma$  are parametrized by the non-smooth midpoint of the piecewise geodesic path, which traces out the equidistant great circle of the two endpoints. We let  $SP_\gamma \subset P_\gamma$  be the closed subfamily consisting of the shorter paths, namely

$$SP_\gamma = \{\tau \in P_\gamma \mid L_0(\tau) \leq \pi\}.$$

If  $\gamma$  is a great semi-circle, define  $P_\gamma$  to be the circular family of great half-circles joining the endpoints of  $\gamma$ , and the subfamily  $SP_\gamma$  to be the family of paths forming either an acute or a right angle with  $\gamma$  at the endpoints. The following lemma is obvious but crucial.

**Lemma 2.2.** *The family  $SP_\gamma$  for a geodesic segment  $\gamma$  with  $L_0(\gamma) = \pi$  is the limit of the families  $SP_{\gamma_i}$  for subarcs  $\gamma_i$  of  $\gamma$  of length tending to  $\pi$ . In fact, if  $\gamma_i$  is any sequence of minimizing geodesic segments converging to  $\gamma$ , then  $SP_{\gamma_i}$  converges to  $SP_\gamma$ .*

Our main technical tool in the next section will be the following result.

**Lemma 2.3.** *If  $f$  is amply negative then there is an  $\epsilon > 0$  so that for all geodesic segments  $\gamma$  with  $\pi - \epsilon \leq L_0(\gamma) \leq \pi$ , there is a path  $\tau \in SP_\gamma$  with  $\int_\tau f ds_0 < -\nu(f)$ .*

*Proof.* If no such  $\epsilon$  exists, then there is a sequence  $\{\gamma_i\}$  with  $L_0(\gamma_i) < \pi$  and  $L_0(\gamma_i) \rightarrow \pi$  such that all  $\tau \in SP_{\gamma_i}$  satisfy  $\int_\tau f ds_0 \geq -\nu(f)$ . There is a convergent subsequence such that  $\gamma'_i(0) \rightarrow \gamma'(0)$  with  $L_0(\gamma) = \pi$ . By Lemma 2.2, the family  $SP_{\gamma_i}$  converges to  $SP_\gamma$ . By the continuity of  $f$ , for every  $\tau \in SP_\gamma$  we have  $\int_\tau f ds_0 \geq -\nu(f)$ , contradicting the assumption that the function  $f$  is amply negative.  $\square$

**Remark 2.4.** Given a piecewise geodesic  $\tau$  over which the integral of  $f$  is negative, we will show in the next section that the length of  $\tau$  decreases under the Guillemin deformation. If, in addition, the curve  $\tau$  has length at most  $\pi$  with respect to the metric  $g_0$ , then the length with respect to the metric  $g_t$  will be shorter than  $\pi$ . That is why we need to work with piecewise geodesics specifically in  $SP_\gamma$ . In order to make the continuity argument above work, one needs to find in each  $SP_\gamma$ , a curve  $\tau$  over which  $f$  integrates negatively. This leads to the amply negative condition we introduced.

### 3. DIAMETER OF GUILLEMIN DEFORMATION

Let  $\Psi^f$  be the conformal factor of the Guillemin deformation, as in Theorem 1.1 above. Thus, the metric  $g_t = \Psi_t^f g_0$  is Zoll, while  $\Psi_0^f = 1$  and  $(d\Psi_t^f/dt)|_{t=0} = f$ . Consider the arclength parametrisation  $\tau(s)$  of a path  $\tau \subset S^2$  for the round metric  $g_0$ .

**Lemma 3.1.** *The energy  $E_t(\tau)$  of a path  $\tau \subset S^2$  for the metric  $g_t$  satisfies*

$$\left. \frac{dE_t}{dt} \right|_{t=0} = \int_{\tau} f \circ \tau \, ds_0.$$

*Proof.* We have

$$\begin{aligned} \frac{d}{dt} E_t(\tau) &= \frac{d}{dt} \int_0^{L_0(\tau)} g_t(\tau'(s), \tau'(s)) ds \\ &= \frac{d}{dt} \int_{\tau} \Psi_t^f \circ \tau \, ds_0 \\ &= \int_{\tau} \left( \frac{d}{dt} \Psi_t^f \right) \circ \tau \, ds_0 \\ &= \int_{\tau} f \circ \tau \, ds_0 \end{aligned}$$

at  $t = 0$ . □

**Proposition 3.2.** *If  $f$  is amply negative, then the associated Guillemin deformation  $g_t = \Psi_t^f g_0$  as in Theorem 1.1 satisfies  $D(g_t) < \pi$  for all sufficiently small  $t > 0$ .*

*Proof.* Denote by  $L_t$  and  $d_t$  the length and the distance with respect to the metric  $g_t$ . Let  $\epsilon > 0$  be chosen as in Lemma 2.3, and let  $\mathcal{A}_\epsilon \subset S^2 \times S^2$  be the set of nearly antipodal pairs, defined by setting

$$\mathcal{A}_\epsilon = \{(p, q) \in S^2 \times S^2 \mid d_0(p, q) \geq \pi - \epsilon\}$$

By continuity, there is a  $\delta > 0$  such that whenever  $0 < t < \delta$ , we have

$$(3.1) \quad d_t(p, q) < \pi \text{ for all } (p, q) \notin \mathcal{A}_\epsilon.$$

Now let  $(p, q) \in \mathcal{A}_\epsilon$ , and  $\gamma$  a minimizing geodesic joining them. Let

$$N(\gamma) = \left\{ \tau \in SP_\gamma \mid \int_{\tau} f \, ds_0 \leq -\nu(f) \right\},$$

and let  $N = \{\tau \in N(\gamma) \mid \pi - \epsilon \leq L_0(\gamma) \leq \pi\}$ . By Lemma 2.3, whenever

$$\pi - \epsilon \leq L_0(\gamma) \leq \pi,$$

the set  $N(\gamma)$  is non-empty. Furthermore, the sets  $N$  and  $N(\gamma)$  are compact. Now for small  $t > 0$ , define a continuous function  $F : N \times \mathbb{R} \rightarrow \mathbb{R}$  by setting  $F(\tau, t) = \frac{dE_t(\tau)}{dt}$ . By Lemma 3.1 and the definition of  $N$ , we have  $F(\tau, 0) \leq -\nu(f)$ . Hence by the compactness of  $N$  and the continuity of  $F$  there is a real  $\delta' > 0$  so that for all  $0 \leq t \leq \delta'$  and all  $\tau \in N$ , we have  $F(\tau, t) < -\frac{1}{2}\nu(f)$ . Therefore the energy given by the expression

$$\int_0^{L_0(\tau)} g_t(\tau'(s), \tau'(s)) ds_0$$

is strictly decreasing in  $t$ . Hence for  $0 < t \leq \delta'$ , it is strictly smaller than the quantity

$$\int_0^{L_0(\tau)} g_0(\tau'(s), \tau'(s)) ds_0 = L_0(\tau).$$

In particular, we obtain for  $0 < t \leq \delta'$ ,

$$\begin{aligned} L_t(\tau) &= \int_0^{L_0(\tau)} \sqrt{g_t(\tau'(s), \tau'(s))} ds_0 \\ &\leq L_0(\tau)^{\frac{1}{2}} \left( \int_0^{L_0(\tau)} g_t(\tau'(s), \tau'(s)) ds_0 \right)^{\frac{1}{2}} \\ &< L_0(\tau). \end{aligned}$$

Thus for each pair  $(p, q) \in \mathcal{A}_\epsilon$  and every  $0 < t \leq \delta'$ , there is a path  $\tau$  from  $p$  to  $q$  with  $L_t(\tau) < L_0(\tau) \leq \pi$ . Hence  $\text{dist}_t(p, q) < \pi$ . Combined with (3.1), this yields the diameter bound  $D(g_t) < \pi$  whenever  $0 < t < \min\{\delta, \delta'\}$ , proving the proposition as well as Theorem 1.4.  $\square$

#### 4. FINE SETS AND THEIR PROPERTIES

Recall that an open hemisphere is an open ball of radius  $\pi/2$  centered at any point of the unit sphere. The construction of amply negative functions in Section 5 exploits *fine* sets, in the following sense.

**Definition 4.1.** A spherical pointset  $X$  is called *fine* if the following three conditions are satisfied:

- (1) no triple of  $X$  is collinear;
- (2) no triple of great circles  $pp'$ , where  $p, p' \in X$ , is concurrent other than at points of  $X$  (as well as their antipodal points);
- (3) every open hemisphere contains at least 3 of the points of  $X$ .

Note that the non-collinearity implies, in particular, that  $X$  contains no pair of antipodal points. Meanwhile, condition (3) implies that at every point of the sphere, there is a Y-like set of tangent directions leading to points of  $X$ .

To see that fine sets exist, start with the set of 4 vertices of the regular inscribed tetrahedron. This gives a set with at least one point in every open hemisphere. We replace each point of the tetrahedron by a generic triple of nearby points. The non-collinearity and non-concurrency follow from genericity, and property (3) follows by construction.

**Definition 4.2.** Given a fine pointset  $X$ , choose  $\epsilon(X) > 0$  such that:

- (1) the closed  $\epsilon(X)$  balls centered at the points of  $X \cup -X$  are disjoint;
- (2) there are at least 3 points of  $X$  in  $B(p, \pi/2 - \epsilon(X))$  for every  $p \in S^2$ .

We note that property (3) of fineness along with standard compactness arguments shows that such a positive  $\epsilon(X)$  exists.

**Lemma 4.3.** *Let  $X$  be a fine set and choose  $\epsilon(X)$  as above. Let  $\Sigma$  the set of unit vectors in  $US^2$  tangent to geodesic segments  $\tau$  of length  $\pi$  satisfying the following two conditions:*

- $\tau(0, \pi) \cap -X = \emptyset$ ;
- $\tau(\epsilon(X), \pi - \epsilon(X)) \cap X \neq \emptyset$ .

*Then  $\Sigma$  is Y-like.*

*Proof.* Fix a unit vector  $v$  at  $p \in S^2$  and let  $\gamma$  be the corresponding geodesic segment of length  $\pi$ . We need to find a  $w \in \Sigma$  at  $p$  making an acute angle with  $v$ . Let  $H$  be the (closed) hemisphere obtained as the union of the  $\tau \in SP_\gamma$ . Then by assumption there are at least three points of  $X$  (call them  $p_1, p_2$ , and  $p_3$ ) in the interior of  $H$  and at a distance greater than  $\epsilon(X)$  from the boundary of  $H$  hence the endpoints of  $\gamma$ . Hence there are at least 3 geodesic segment  $\tau_1, \tau_2$ , and  $\tau_3$  in the interior of  $SP_\gamma$  passing through the points  $p_i, i = 1, 2, 3$ . If two of these paths coincide (say  $\tau_1 = \tau_2$ ) then  $\tau_1$  passes through  $p_1$  and  $p_2$  so the initial point  $p$  of  $\gamma$  is not in  $X$  and  $\tau_1$  avoids  $-X$  (by condition 1 of being fine). If all of these paths are pairwise distinct and also pass through points of  $-X$  (say  $-p_4, -p_5$ , and  $-p_6$  respectively) then the initial point  $p$  of  $\gamma$  would lie on the 3 great circles  $p_1p_4, p_2p_5, p_3p_6$  which contradicts either condition 2 (if  $p \notin X \cup -X$ ) or condition 1 (if  $p \in X \cup -X$ ). Thus we see that there is a  $\tau$  in the interior of  $SP_\gamma$  containing an element of  $X$  at least  $\epsilon(X)$  from the endpoints, and no element of  $-X$  in its interior. The tangent vector of  $\tau$  is the  $w$  we seek.  $\square$

## 5. EXISTENCE OF AMPLY NEGATIVE FUNCTIONS

The goal of this section is to prove the following proposition.

**Proposition 5.1.** *There exist amply negative functions.*

We will construct such functions by defining odd functions that approximate the sum of  $\pm\delta$  (Dirac delta) functions centered at points of  $-X$  and  $X$  for a fine set  $X$ . For our approximate  $\delta$  functions we take for each  $p \in S^2$  the smooth function  $\delta_p^\epsilon$  with support included in the ball  $B(p, \epsilon)$  with

$$\delta_p^\epsilon(q) = \exp(1/\epsilon) \cdot \exp\left(\frac{1}{d(p, q) - \epsilon}\right)$$

for  $q \in B(p, \epsilon)$ .

We will use the following (nearly obvious) lemma.

**Lemma 5.2.** *if  $\gamma$  is a diameter of  $B(p, \epsilon)$  (i.e. geodesic through the center of length  $2\epsilon$ ) and  $\tau$  is any geodesic segment in  $B(p, \epsilon)$  then  $\int_\tau \delta_p^\epsilon \leq \int_\gamma \delta_p^\epsilon$  with equality holding if and only if  $\tau$  is also a diameter.*

*Proof.* To see this (since  $\delta_p^\epsilon \geq 0$ ) we can assume (by extending  $\tau$  if needed) that  $\tau$  runs from a boundary point to a boundary point and has length  $2l < 2\epsilon$ . Since for  $t \leq l$  we have  $d(p, \tau(t)) \geq \epsilon - t = d(p, \gamma(t))$  we have

$$\int_\tau \delta_p^\epsilon = 2 \int_0^l \delta_p^\epsilon(\tau(t)) dt \leq 2 \int_0^l \delta_p^\epsilon(\gamma(t)) < 2 \int_0^\epsilon \delta_p^\epsilon(\gamma(t)) = \int_\gamma \delta_p^\epsilon.$$

□

We are now ready to define our functions.

**Definition 5.3.** For  $\epsilon(X) > \epsilon > 0$  set

$$f_X^\epsilon = \sum_{p_i \in X} (\delta_{-p_i}^\epsilon - \delta_{p_i}^\epsilon)$$

Note that  $f_X^\epsilon$  is a smooth odd function. We will now prove that for sufficiently small  $\epsilon > 0$  the function  $f_X^\epsilon$  is amply negative.

**Lemma 5.4.** *For every  $v \in US^2$  there is an  $\epsilon(v)$  with  $\epsilon(X) > \epsilon(v) > 0$  and an open neighborhood  $U(v)$  of  $v$  in  $US^2$  (note that the base point also varies) such that for all  $w \in U(v)$  there is a geodesic segment  $\tau$  of length  $\pi$  whose initial tangent vector makes an acute angle with  $w$  (hence it starts at the base point of  $w$ ) and*

$$\int_\tau f_X^\epsilon < 0$$

for all  $\epsilon(v) > \epsilon > 0$ .

Note that the  $\epsilon(v) > 0$  we find in the proof below will tend to 0 as the base point of  $v$  tends to  $X$  (while not being in  $X$ ). This turns out not to be a problem since by the compactness of  $US^2$  a finite number  $U(v_i)$  cover  $US^2$  and hence we can take any  $\epsilon$  less than the smallest of the  $\epsilon(v_i)$  and have that  $f_X^\epsilon$  is amply negative. This therefore proves Proposition 5.1.

*Proof of Lemma 5.4.* To prove the lemma, we consider two cases. First assume that the base point of  $v$  is not in  $X \cup -X$ . Let  $\tau$  (whose existence is promised in Lemma 4.3) be a geodesic segment of length  $\pi$  making an acute angle with  $v$  that misses  $-X$  and passes through at least one  $p \in X$  that has distance greater than  $\epsilon(X)$  from its endpoints. Thus we can choose  $\epsilon(v)$  so small that  $\tau$  misses  $B(q, 2\epsilon(v))$  for all  $q \in -X$ . Now for  $w$  in a small enough neighborhood  $U$  of  $v$ , let  $\bar{\tau}$  be the geodesic segment of length  $\pi$  through the base point of  $w$  and  $p$ . For small enough  $U$ ,  $\bar{\tau}$  will still miss all the  $B(q, \epsilon(v))$  for  $q \in -X$  while  $\bar{\tau}$  will make an acute angle with  $w$ . Thus, for  $\epsilon(v) \geq \epsilon > 0$ , we have  $f_X^\epsilon \leq 0$  along  $\bar{\tau}$  and is negative near  $p$  so we see  $\int_{\bar{\tau}} f_X^\epsilon < 0$ .

In the second case the basepoint  $p_0$  of  $v$  is in  $X \cup -X$ . We will assume  $p_0 \in -X$  since the other case is the same (by reversing orientation of all geodesics). Note that any  $\tau$  making an acute angle with  $v$



which intersects  $X$  in its interior cannot also intersect  $-X$  in its interior by property 1 of a fine set. So there are two (in fact three) geodesic segments  $\tau_1$  and  $\tau_2$  in the interior of  $SP_\gamma$  that pass through a  $p_1$  and a  $p_2$  respectively and no element of  $-X$  in its interior and such that  $p_1$  and  $p_2$  have distance greater than  $\epsilon(X)$  from  $p_0$  and  $-p_0$ . Now we choose  $\epsilon(v) > 0$  so small that for all  $q \in -X$  and  $q \neq p_0$ ,  $B(q, 2\epsilon(v))$  miss both  $\tau_1$  and  $\tau_2$ . Again for  $w$  in a small neighborhood  $U$  of  $v$  let  $\bar{\tau}_1$  (resp  $\bar{\tau}_2$ ) be the geodesic segment of length  $\pi$  starting at the basepoint of  $w$  and passing through  $p_1$  (resp.  $p_2$ ).

If  $U$  is small enough we can assume both that  $\tau_i$  make acute angles with  $w$  and that for all  $q \in -X$  with  $q \neq p_0$ ,  $\bar{\tau}_1$  and  $\bar{\tau}_2$  miss  $B(q, \epsilon(v))$ . Along  $\bar{\tau}_1$  (resp.  $\bar{\tau}_2$ ) we have, for  $\epsilon(v) \geq \epsilon > 0$ ,  $f_X^\epsilon \leq 0$  except on

$$\bar{\tau}_1 \cap B(p_0, \epsilon)$$

(respectively, on  $\bar{\tau}_2 \cap B(p_0, \epsilon)$ ). Both  $\bar{\tau}_1 \cap B(p_0, \epsilon)$  and  $\bar{\tau}_2 \cap B(p_0, \epsilon)$  cannot be diameters since that would put  $p_0$ ,  $p_1$  and  $p_2$  on the same great circle (namely the one through  $p_0$  and the base point of  $w$ ). So assume  $\bar{\tau}_1 \cap B(p_0, \epsilon)$  is not a diameter. Then since we know on the other hand that  $\bar{\tau}_1 \cap B(p_1, \epsilon)$  is a diameter, Lemma 5.4 tells us that  $\int_{\bar{\tau}_1} f_X^\epsilon < 0$ .  $\square$

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SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, SUISSE  
*E-mail address:* `florent.balacheff@math.unige.ch`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104-6395 USA  
*E-mail address:* `ccroke@math.upenn.edu`

MIKHAIL G. KATZ, DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN 52900 ISRAEL  
*E-mail address:* `katzmik ‘‘at’’ math.biu.ac.il`